UNIVERSITY OF LONDON [MP1 2003]

B.Sc. and M.Sci. DEGREE EXAMINATIONS 2003

For Internal Students of Imperial College London

This paper is also taken for the relevant examination for the Associateship of the City and Guilds of London Institute

FIRST YEAR STUDENTS OF PHYSICS

MATHEMATICS - M. PHYS 1

Date: Thursday 1st May 2003 Time: 10 am - 1 pm

Do not attempt more than SIX questions

[*Before starting, please make sure that the paper is complete; there should be 5 pages, with a total of 10 questions. Ask the invigilator for a replacement if your copy is faulty.*]

[MP1 2003]

1. (i) Find the limit:

$$\lim_{x\to\infty} x^2\left[(1+x^4)^{1/2} - x^2\right],$$

(ii) Using L'Hôpital's theorem, or otherwise, show that

$$\lim_{x\to 1}\frac{x^n+mx-(m+1)}{x^m+nx-(n+1)} = \begin{cases} 1 & , \quad m+n\neq 0\,, \\[1ex] \dfrac{n-1}{n+1} & , \quad m+n=0\,. \end{cases}$$

What happens in the case $n = -m = -1$?

(iii) Differentiate, from first principles, the function

$$y = \tan x\,.$$

2. (i) Using integration by parts, or otherwise, evaluate the indefinite integral

$$\int x\sin^{-1}x^2 dx\,.$$

(ii) Consider a semi-circular disc of radius a centred at the origin and occupying the region $y>0$. Now cut away the two strips with $|\,x\,|>a/2$ to leave the region below

Find the area A and the position of the centre of mass $(\overline{x}, \overline{y})$.

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3. Find the four stationary points of the function

$$u(x,\,y) \;=\; xy(x+y-2)$$

and identify their nature.

Draw a sketch in the $x-y$ plane showing these points and the lines on which $u=0$. Indicate the regions in which u is positive and in which u is negative.

4. (i) Point A has coordinates $(3,\,-1,\,0)$, Point B has coordinates $(-1,\,5,\,2)$. Find the unit vectors in the following directions:

   (a) from the origin to A;

   (b) from B to the origin;

   (c) from A to B.

   (ii) Write down the vector equation of the straight line which passes through A and B. Hence find the Cartesian equations of this line.

5. (i) For $z=2i$ calculate the real and imaginary parts of the three values of $z^{1/3}$.

   (ii) Sketch the locations of the three values of $z^{1/3}$ in the complex plane.

   (iii) For $z=2i$ calculate the real and imaginary parts of one value of $\ln z$.

6. (i) Find the eigenvalues λ_1 and λ_2, and normalized eigenvectors $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$, of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -2 & -1 \end{pmatrix} .$$

(ii) Write down the matrix $\mathbf{B}$ which has the normalized eigenvectors of matrix $\mathbf{A}$ as its columns, i.e.,

$$\mathbf{B} = \begin{pmatrix} \hat{s}_{1x} & \hat{s}_{2x} \\ \hat{s}_{1y} & \hat{s}_{2y} \end{pmatrix} .$$

A matrix formed from the normalized eigenvectors of a real symmetric matrix, such as $\mathbf{A}$, has the property that its inverse is equal to its transpose. Check that $\mathbf{B}^{-1} = \mathbf{B}^T$.

(iii) Show by multiplying out the matrices that

$$\mathbf{B}^{-1}\,\mathbf{AB} = \mathbf{C}\ ,$$

where

$$\mathbf{C} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} .$$

7. State the divergence theorem, identifying the quantities that are being integrated and the regions over which the integrations are carried out.

Consider the vector field

$$\mathbf{V} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} .$$

Find the outward flux across the boundary of the region

$$1 \le x^2 + y^2 + z^2 \le 4$$

both by using the divergence theorem and by directly calculating the flux across the boundary and thereby verify the divergence theorem.

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8. Consider the vector field

$$\boldsymbol{V} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k} \ ,$$

and a cylindrical surface whose base is a circle of radius R centered at the origin in the x-y plane and which has a height h, i.e. its top surface lies on the plane $z = h$.

(i) Evaluate

$$\oint \boldsymbol{V} \cdot d\boldsymbol{r}$$

counter-clockwise (looking from the positive z-direction) around the circle $x^2 + y^2 = R^2$ in the x-y plane.

(ii) Evaluate

$$\int\int (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S}$$

over the surface (top and sides) of the cylinder.

(iii) Evaluate

$$\int\int \nabla \times \boldsymbol{V}) \cdot d\boldsymbol{S}$$

over the circular region in the x-y plane bounded by $x^2 + y^2 = a^2$.

The unit normal to the region points in the positive z direction.

(iv) What do your results in (i), (ii) and (iii) say about (a) Stokes' theorem, and (b) the significance of surfaces in (ii) and (iii).

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9. Consider the differential equation for a forced undamped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F_0 \cos \omega t \ , \tag{1}$$

where $x(t)$ is the displacement of the oscillator at time t, ω_0 is the natural frequency of the oscillator, and the forcing is carried out with amplitude F_0 and frequency ω.

(i) Show that the general solution of the corresponding homogeneous equation is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \ .$$

where A and B are constants to be determined.

(ii) Find a particular solution of (1) and, hence, show that, provided $\omega^2 \neq \omega_0^2$, its general solution can be written as

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t) \ .$$

(iii) Obtain the solution with the initial conditions $x(0) = 0$ and $x'(0) = 0$, i.e. the oscillator is initially at rest, and show that it can be expressed in the form

$$x(t) = \frac{2F_0}{\omega_0^2 - \omega^2} \sin\left[\frac{1}{2}(\omega_0 - \omega)t\right] \sin\left[\frac{1}{2}(\omega_0 + \omega)t\right] \ .$$

(iv) Consider the solution in (iii) as $\omega \to \omega_0$. By using L'Hôpital's rule, or otherwise, show that the solution in this limit becomes

$$x(t) = \frac{F_0 t}{2\omega_0} \sin(\omega_0 t) \ .$$

What is the behaviour of this solution as $t \to \infty$?

10. Consider the equations of motion for $\boldsymbol{u} = (u_x,\, u_y,\, u_z)$ given by

$$\frac{du_x}{dt} \;=\; u_y\,, \qquad \frac{du_y}{dt} \;=\; -\,u_x\,, \qquad \frac{du_z}{dt} \;=\; 0\,.$$

(i) Show that these equations imply that both u_x and u_y satisfy the following equation:

$$\frac{d^2u}{dt^2} \;+\; u \;=\; 0\;,$$

where $u = u_x$ or $u = u_y$. Find the general solution of this equation.

(ii) Use the original equations for u_x and u_y to obtain the solutions in terms of the initial values $u_x(0)$ and $u_y(0)$. Hence, show that the solution for $\boldsymbol{u}$ can be written as

$$\begin{aligned} u_x(t) &= u_x(0)\cos t + u_y(0)\sin t\;, \\ u_y(t) &= u_y(0)\cos t - u_x(0)\sin t\;, \\ u_z(t) &= u_z(0)\;. \end{aligned}$$

(iii) if $\boldsymbol{u}$ is the velocity of a particle, what is the trajectory in the case where

(a) $u_z(0) = 0$ and (b) $u_z(0) \neq 0$?

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